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Longitudinal oscillations in a classical Yang-Mills plasma

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Abstract. An analytical approach to the non linear system describing classical 1D longitudinal oscillations in an SU(2) plasma is proposed. A physical interpretation is given for two distinct "Abelian" and "non-Abelian" phases of the non-linear oscillations. Numerical estimates of meaningful physical quantities are made. Analytical results are extended to the physically more relevant SU(3) case, for which numerical simulations have been also carried out.

PACS. 12.38.Mh Quark-gluon plasma – 52.35.Mw Nonlinear waves and nonlinear wave propagation (including parametric effects, mode coupling, ponderomotive effects, etc.)

1 Introduction

In this paper we discuss a fluid model for a SU(2) Yang-Mills plasma. This model is useful to understand, in a relatively simple framework, some aspects of the phenomenology of physical systems in which quark-gluon plasma effects play an important role. It is well-known that quark-gluon plasmas present a complex phenomenology [1,2] which has been extensively investigated by means of quantum field theoretical techniques, such as lattice QCD simulations at finite temperature [1,2]. However, it is also commonly assumed that collective phenomena of essentially classical nature occur in quark-gluon plasmas [1], such as the screening mechanism, which is in competition with the well-known anti-screening effect due to vacuum polarization in zero temperature QCD. The existence of such collective phenomena justifies a classical approach to quark-gluon plasmas. Starting from the work of Wong [3], Kajantie and Montonen [4], we show that in the system of fluid equations they derive, conservation laws can be written in a general and gauge invariant form, and we also discuss the physical meaning of quantities involved in such equations. We further investigate the system obtained by Bhatt *et al.* [5] and Kaw [6], in the case of pure longitudinal 1D plasma oscillations, and propose a physical interpretation of the phenomenology observed by these authors. We also present an analytical approach providing an approximate solution of this kind of system, showing that the main physical features may be predicted and interpreted in this framework. Finally, we extend our analytical solution to the case of an SU(3) plasma, and we perform numerical simulations of the analogue of the system found by Bhatt *et al.* [5]. We find that the precession of the color charge vector in color space gives rise to a dynamically decoupled phase (on the fast time-scale of plasma oscillations) both in the SU(2) and the SU(3) case, which may be predicted and investigated by means of the analytical approach proposed here.

In the present work, we employ the "cold plasma" approximation, *i.e.*, we assume that $\hbar\omega_{\rm p} \gg kT$, where $\omega_{\rm p}$ is the plasma frequency and T the temperature. More specifically, all physical quantities will be computed in the cold plasma limit, and the corresponding expressions will be valid only in this limit; *e.g.*, this will apply to the expression of the plasma frequency, reported in Section 3.

In addition we also restricted ourselves to the case $mc^2 \gg \hbar\omega_{\rm p} \gg kT$. This cold non relativistic limit is discussed in reference [7] and used there as a physically relevant one for the $q\bar{q}$ pair production phenomenology in a quark-gluon plasma.

The opposite limit, $\hbar\omega_{\rm p} \ll kT$, is analyzed by Braaten and Pisarski [8], where the plasmon damping rate due to thermal gluon interactions is computed within a perturbative ($g \to 0$) framework.

Here, it is worth emphasizing that the characteristic times for the onset of collective non-Abelian effects, that we discuss in the present paper, are of the same order or smaller than the computed characteristic plasmon damping times [8]. However, the present assumption that plasmon interactions with thermal gluons are negligible does not qualitatively alter our primary result that there exist two different time scales: a short one, naturally associated with $\omega_{\rm p}^{-1}$, and a long one, due to collective screening of the color charge. This point is particularly relevant for the SU(3) case, where the long time scale dynamics becomes rapidly the most relevant one, as it will be apparent from the analysis of Section 8. In fact, the results of the present work suggest the possibility of simultaneously

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handling both collective non-Abelian effects and plasmon interactions with thermal gluons by extending the present multiple time scale analytical approach. This, however, is beyond the scope of the present analysis. The comparison between characteristic time scales associated with collective screening of the color charge and plasmon thermal damping times will be further addressed in the conclusions (Sect. 9).

2 Basic plasma equations and conservation laws

Following Kajantie and Montonen [4] we describe classical particles by three fields: density n, velocity $\mathbf{v}$, and color charge vector I_a $(a = 1, 2, 3)$. The plasma particles generate a four current density which acts as a source of classical Yang-Mills fields A_a^μ. The basic equations for the colored particles in external gauge fields, obtained by Kajantie and Montonen [4], are:

$$\frac{\partial}{\partial t}\mathbf{v}_A + (\mathbf{v}_A \cdot \boldsymbol{\nabla})\mathbf{v}_A = \frac{g}{m_A} I_{Aa}(\mathbf{E}_a + \mathbf{v}_A \times \mathbf{B}_a), \quad (2.1)$$

$$\frac{\partial}{\partial t} n_A + \boldsymbol{\nabla} \cdot (n_A \mathbf{v}_A) = 0, \quad (2.2)$$

$$\frac{\partial}{\partial t} I_{Aa} + (\mathbf{v}_A \cdot \boldsymbol{\nabla}) I_{Aa} = -g\varepsilon_{abc}(A_b^0 - \mathbf{A}_b \cdot \mathbf{v}_A) I_{Ac}. \quad (2.3)$$

By means of particle fields, it is possible to define [3,4] the four current J_a^μ as:

$$J_a^0 = g \sum_A n_A I_{Aa}, \quad (2.4)$$

$$\mathbf{J}_a = g \sum_A n_A \mathbf{v}_A I_{Aa}, \quad (2.5)$$

where the index A indicates an SU(2) doublet of mass m_A. In the following, we will consider a plasma with only two doublets, of mass m_1 and m_2 respectively, and with $m_2 \gg m_1$.

This assumption implies that heavy colored particles have no dynamics and form only a background which ensures global particle charge neutralization at equilibrium [5,6], *i.e.* $I_{10a} + I_{20a} = 0$, and $I_{2a} = const = I_{20a}$. Gauge fields, generated by colored particles, are described by the tensor:

$$F^{\mu\nu} = F_a^{\mu\nu} T_a = \frac{\mathrm{i}}{g}[D^\mu, D^\nu], \quad (2.6)$$

where T_a are the infinitesimal generators of an SU(2) transformation, whose commutation rules are:

$$[T_a, T_b] = \mathrm{i}\varepsilon_{abc} T_c, \quad (2.7)$$

D^μ is the covariant derivative :

$$D^\mu = \partial^\mu - \mathrm{i} g A_a^\mu T_a \quad (2.8)$$

and

$$F_A^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\varepsilon_{abc} A_b^\mu A_c^\mu. \quad (2.9)$$

For the four current J_a^μ, generated by the particles, there exists a continuity equation:

$$[D_\mu, J^\mu] = 0 \quad (2.10)$$

which, projected on the SU(2) axes, becomes:

$$\partial_\mu J_a^\mu = g\varepsilon_{abc} A_{\mu b} J_c^\mu = 0. \quad (2.11)$$

Ampère-Maxwell equations for the gauge fields are:

$$[D_\mu, F^{\mu\nu}] = J^\nu, \quad (2.12)$$

i.e.,

$$\partial_\mu F_a^{\mu\nu} + g\varepsilon_{abc} A_{\mu b} F_c^{\mu\nu} = J_a^\nu. \quad (2.13)$$

We briefly discuss gauge transformation properties [5] of equations and quantities introduced above. These properties are, in general, well-known. Nonetheless, we find it useful to briefly recall them here. Moreover, these properties will allow us to find gauge invariant terms which are also conserved quantities.

If U is a matrix which represents a gauge transformation:

$$U = \mathrm{e}^{\mathrm{i} g \varepsilon_a(x) T_a}, \quad (2.14)$$

where $\varepsilon_a(x)$ are the infinitesimal parameters of the gauge transformation, and the covariant derivative transforms as:

$$D_\mu \to U D_\mu U^+, \quad (2.15)$$

from its definition in terms of covariant derivative, it is well-known that $F^{\mu\nu}$ also transforms covariantly:

$$F_{\mu\nu} \to U F_{\mu\nu} U^+. \quad (2.16)$$

It follows that the left hand side of Ampère's equation transforms covariantly, so that the four current also behaves as a covariant quantity under a gauge transformation [5]:

$$J_\mu \to U J_\mu U^+. \quad (2.17)$$

From the definition of the four current in terms of density, velocity and color charge vector, one has that, since n_A and $\mathbf{v}_A$ are scalar quantities in color space, the color charge vector must have the gauge transformation properties of the four current [5]:

$$I_A \to U I_A U^+. \quad (2.18)$$

Non-Abelian gauge theories [9], as SU(2), are constructed by representing particles with multiplets in the color space, which is also the charge space, and by imposing local invariance of the theory under rotations in this space. In SU(2), particles are represented by doublets while the color charge vector, as an operator over particle states, acts in the color space as an angular momentum, which is also the generator of SU(2) rotations. The three component I_a of the color charge vector are indeed, from quantum-mechanical point of view, the average values over the particle state of the three generators of infinitesimal rotations in SU(2) space, T_a, and represent the coupling strength to the gauge fields, which are also represented by a SU(2) vector A_a^μ. While in Abelian theories the charge is a quantum number associated to the particle, which is not varied by the interaction, in non-Abelian theories the color charge vector is a dynamical variable that, due to interaction, evolves in time. Interacting gauge fields and particles exchange color charge, so that only the global charge, *i.e.* the sum of the gauge field charge and of the particle charge, is conserved: continuity equation (2.10) expresses this property. The interaction Hamiltonian:

$$H_{\rm int} = g\gamma_\mu A_a^\mu T_a, \tag{2.19}$$

represents the coupling of two angular momenta in the SU(2) color space: the gauge fields and the particle color charge vector. Here, γ_μ are the Dirac matrices [10]. The evolution equations of the mean values of physical quantities are obtained by considering the mean value of the Heisenberg equations for the time evolution of the operators representing such quantities (Ehrenfest theorem). In our case, the color charge vector evolution equation is:

$$\langle \frac{\mathrm{d}}{\mathrm{d}t} T_a \rangle = \mathrm{i}\langle [T_a, H_{\rm int}]\rangle. \tag{2.20}$$

Taking the non relativistic limit of equation (2.20) we get equation (2.3). We stress that equation (2.3) can be obtained also by using equation (2.2) and the continuity equation (2.11).

In the case of fluid equations, both the particle velocity and the color vector charge are quantities that are also averaged over a large number of particles. This average is performed over a volume which is small with respect to the wavelength of plasma oscillations but still contains a very large number of fluid particles.

The gauge invariant equation:

$$\mathrm{Tr}\big(F_\nu^0\,[D_\mu, F^{\mu\nu}]\big) = \mathrm{Tr}(F_\nu^0 J^\nu) \tag{2.21}$$

represents, in differential form, the generalized Poynting theorem; this equation describes the energy balance between gauge fields and particles. The right side of equation (2.21) represents the density of mechanical work done by the gauge fields on the particles ($\mathbf{E}_a \cdot \mathbf{J}_a$). By integrating equation (2.21) with respect to time and over the whole volume, we obtain the energy conservation law.

The gauge invariant equation:

$$\mathrm{Tr}\,(I\,[D_\mu, J^\mu]) = 0 \tag{2.22}$$

considered for simplicity for one doublet only, gives us, by using equation (2.2):

$$\frac{1}{2}\frac{\partial}{\partial t}(I_a I_a) + I_a \mathbf{v}\cdot\boldsymbol{\nabla} I_a = 0. \tag{2.23}$$

This equation expresses the fluid local conservation of the color charge vector norm.

3 Longitudinal oscillations in an SU(2) plasma

Following Bhatt *et al.* [5], we restrict ourselves to the study of equations (2.1–2.3) and the Ampère's equation (2.13) in the case of 1D longitudinal waves, $A_a^{1,2} = 0$, propagating along the z-axis, with all quantities assumed to be dependent on spatial and time coordinates only through the variable $\xi = z + \beta t$, where β is the wave phase velocity. In such a way, making the gauge choice $A_a^0 = 0$, our starting point is the non linear system of equations found by reference [5]:

$$A_a^{3''} = \beta^{-2}\sum_A g n_A I_{aA} v_A, \tag{3.1}$$

where primes indicate the derivative with respect to ξ. Equation (3.1) may be rewritten in the following form, involving only dimensionless quantities [5]:

$$\ddot{a}_a = -\left(\sum_b \delta_{ab} + \varepsilon\varepsilon_{abc} a_b \dot{a}_c\right)\sum_d a_d. \tag{3.2}$$

Here $a_a = A_a^3/a_0$, with $a_0 \equiv \omega_{\rm p}/g$. Furthermore, $\varepsilon = g^2 I_0 a_0^2/m_1\omega_{\rm p}$ is a parameter which measures the non-Abelian coupling strength, $\omega_{\rm p}^2 = g^2 n_0 I_0^2/m_1$ is the frequency of plasma oscillations, n_0 is the equilibrium density of both light and heavy quarks, $I_0 \equiv I_{10a}$ for $a = 1,2,3$, while dots indicate the derivative with respect to the dimensionless variable $\widehat{t} = \omega_{\rm p}\xi/\beta$. From now on we indicate $\widehat{t}$ with the symbol t, both for simplicity and to maintain the same notation of Bhatt *et al.* [5].

This non linear system describes two kinds of motion with different time scales: the linear term represents the same oscillations as those of an U(1) (Abelian) plasma. Abelian oscillations occur on a fast time scale with plasma frequency $\omega_{\rm p}$. Meanwhile, the non linear term expresses the non-Abelian coupling between the gauge fields and the particles color charge vector, which are angular momenta in the color space. This last term, which describes a sort of "precession" in SU(2), is responsible of the evolution of the system on a time scale which is slow with respect to that of plasma oscillations.

The SU(2) vector $\dot{\mathbf{a}}_a$ rotates in the color space due to the term $\varepsilon(\mathbf{a}\times\dot{\mathbf{a}})v$, where v fluid velocity given, in the present variables, by the term [5] $\sum_b a_b$.

Since the color charge vector of the light particle, in these variables, is:

$$I_a = I_0\Big(\sum_b \delta_{ab} + \varepsilon\varepsilon_{abc} a_b \dot{a}_c\Big) = I_{0a} + \Delta I_a, \tag{3.3}$$

one can see that the "thrust" driving the rotation of $\dot{\mathbf{a}}_a$ is proportional to the variation ΔI_a of the color charge vector.

The system (3.2) may be re-written in a more natural way [5] by means of the following change of variables:

$$\begin{cases} x_1 = (a_1 + a_2 + a_3) \\ x_2 = \sqrt{3/2}(a_1 - a_3) \\ x_3 = \sqrt{1/2}(a_1 - 2a_2 + a_3) \end{cases}. \tag{3.4}$$

This orthogonal transformation represents, except for a normalization factor, a rotation plus an axis inversion in the color space ($a_a a_a = x_i x_i/3$).

In these variables, the system (3.2) becomes:

$$\begin{cases} \ddot{x}_1 = -(\omega_{\rm p}^2 - \varepsilon_0 M_1)x_1 \\ \ddot{x}_2 = \varepsilon_0 M_2 x_1 \\ \ddot{x}_3 = \varepsilon_0 M_3 x_1, \end{cases} \quad . \tag{3.5}$$

where $M_i = \varepsilon_{ijk} x_j \dot{x}_k$, $\varepsilon_0 = \varepsilon/\sqrt{3}$, and $\omega_{\rm p} = \sqrt{3}$ is a dimensionless plasma frequency, which, for simplicity, is still indicated as $\omega_{\rm p}$.

For the system (3.5) two conservation laws hold, which can be easily derived from the gauge invariant equations (2.21, 2.22) respectively:

$$\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + \frac{\omega_{\rm p}^2}{2} x_1^2 = E \tag{3.6}$$

which represents total energy conservation, and

$$\frac{\varepsilon_0}{\omega_{\rm p}^2}(M_1^2 + M_2^2 + M_3^2) - 2M_1 = 0, \tag{3.7}$$

which expresses the conservation of the norm of the color charge vector.

In other words, the conservation equation (3.7) reformulates, in the variables of Bhatt *et al.* [5], the equation:

$$I_a I_a = const = I_{0a} I_{0a}. \tag{3.8}$$

4 Qualitative behavior of longitudinal oscillations in an SU(2) plasma

In the present work, we propose a physical picture of the dynamic behavior of the non-linear system (3.5), already observed in the numerical studies of Bhatt *et al.* [5]. Furthermore, we discuss an analytical solution of this system, which provides better insight into the dynamic processes it involves. Our analytical approach, based on asymptotic techniques [11], will be used to demonstrate the existence of a bistable behavior, already found in the numerical solution by Bhatt *et al.* [5]. In these cases, anticipating the numerical results reported in Section 7, there is a sharp transition from a phase characterized by oscillations that are quite similar to those of an Abelian plasma (in the following we will refer to this phase as the Abelian or the plasma phase) to another phase, which we will call non-Abelian, characterized by faster oscillations of smaller amplitude with non-zero mean value (see Fig. 1).

In order to further discuss the physical implications of such behavior, we recall that the totally symmetric linear combination of the gauge fields, x_1, also represents the fluid velocity of the particles [5]. Meanwhile it is easy to see that, in the right hand side of the first equation of the system (3.5), the term $\omega_{\rm p}^2 - \varepsilon_0 M_1$ is proportional to $\sum_a I_a$, *i.e.* the sum of the components of the color charge vector. So M_1 may be identified with $\sum_a \Delta I_a$. As already

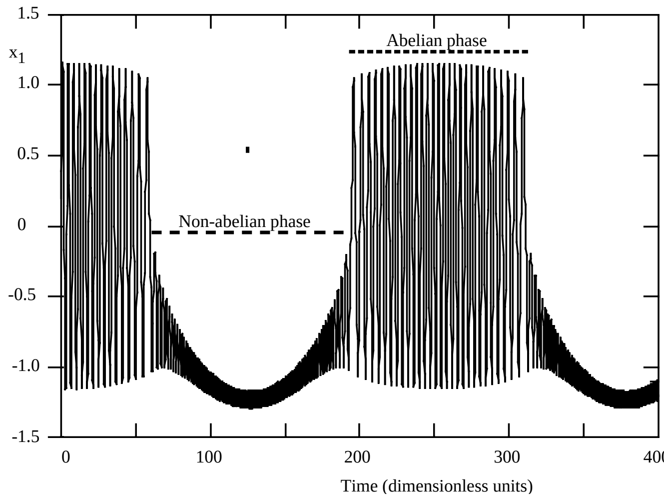

Fig. 1. Abelian and non-Abelian oscillations in an SU(2) quark-gluon plasma exhibiting bistable behavior. The function x_1, representing the light quark fluid velocity, is plotted *versus* the dimensionless time.

mentioned, numerical solutions show that, for sufficiently small values of the parameter ε_0, the system oscillates between two types of behavior, in the present work called the Abelian and non-Abelian phases. In the non-Abelian phase, fast oscillations are superimposed on a slower time evolution and their amplitude is much smaller than their mean value. Thus, if we remember that x_1 represents the fluid velocity of the light quarks, the non-Abelian phase may be interpreted as a decoupling phase, due to the precession of the color charge vector, in which the light quarks drift freely, sliding with respect to the heavy quark background with an approximately constant fluid velocity, while in the Abelian phase they oscillate around this background with a zero mean velocity, as in the case of ordinary plasma waves in an electromagnetic plasma. This decoupling leads to an increase of the physical space accessible to the light particles.

As discussed in Section 7, this drift phenomenon may be effectively represented by particle phase space diagrams (see Fig. 2), in which the particle velocity x_1 is represented as function of the position, *i.e.* the x_1 integral over the time.

If we interpret the transition to the non-Abelian phase in terms of a decoupling on the slow time scale, it is natural to expect that this may happen when the following condition is verified:

$$\left\langle \sum_a I_a \right\rangle = 0, \tag{4.1}$$

where $\langle\ \rangle$ indicates an average over the fast time scale (that of plasma oscillations). The condition (4.1) means that, during the non-Abelian phase, the right hand side of the first equation of the (3.5) system averages to zero. In terms of M_1 the condition (4.1) implies that M_1 saturates to the value:

$$\langle M_1 \rangle = \frac{\omega_{\rm p}^2}{\varepsilon_0} \, . \tag{4.2}$$

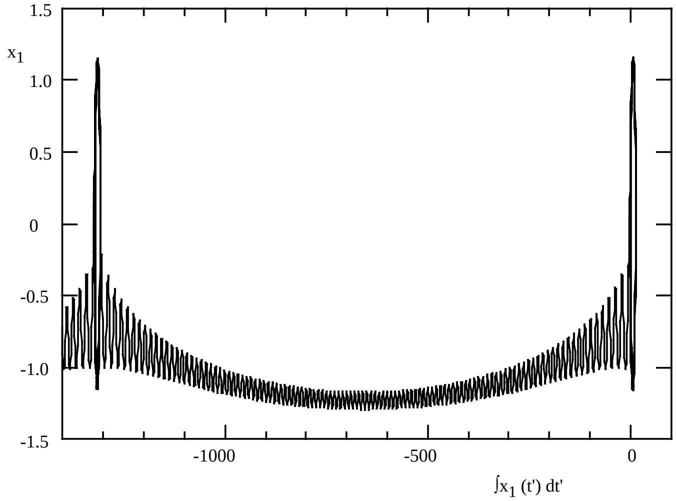

Fig. 2. Drift in the physical space of the light quarks during the non-Abelian phase of plasma oscillations. The fluid velocity x_1 is plotted *versus* the physical space (the integral of x_1). The squeezed vertical ellipses represent the Abelian phase orbits.

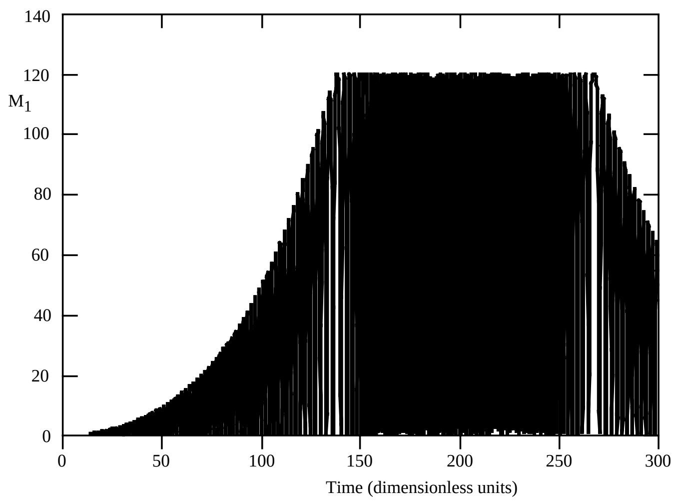

Fig. 3. Time evolution of the function M_1, showing the saturation of $\langle M_1\rangle$ at the transition to the non-Abelian phase.

This saturation value of M_1 is kept constant during all the non-Abelian phase. In Figure 3 we show the numerical solution of M_1, from which it is evident that the transition to the non-Abelian phase is accompanied by a saturation of M_1, which, during this phase, oscillates with amplitude $\omega_{\mathrm{p}}^2/\varepsilon_0$ around a constant mean value given indeed by equation (4.2). This saturation value can also be obtained from the conservation of the norm of the color charge vector, equation (3.7). When M_2 and M_3 become negligible with respect to M_1, equation (3.7) gives the maximum M_1 value:

$$M_{1\max} = \frac{2\omega_{\mathrm{p}}^2}{\varepsilon_0}\cdot \tag{4.3}$$

After the above considerations, we may now proceed to an analytical study of the system (3.5).

5 Analytical approach to the nonlinear system

Let us study the system (3.5) using multiple scale analysis methods [11].

We can introduce two time scales t and $\tau \approx \varepsilon_0 t$; as we will see, the fast scale is that of plasma oscillations, while the slow scale describes the rotations of the color charge vector I_a in the color space due to the coupling with the charged Yang-Mills SU(2) fields.

Expanding all quantities in asymptotic series of ε_0, we have that the derivative with respect to the fast scale t is of order 1, while the derivative with respect to the slow scale τ is of order ε_0:

$$\partial_t \sim 1,$$
$$\partial_\tau \sim \varepsilon_0.$$

Total time derivatives formally become:

$$\mathrm{d}/\mathrm{d}t = \partial_t + \partial_\tau,$$
$$\mathrm{d}^2/\mathrm{d}t^2 = \partial_t^2 + 2\partial_\tau\partial_t + \partial_\tau^2.$$

The optimal ordering of the x_i fields for a non trivial solution is:

$$x_1 \sim 1,$$
$$x_2 \sim x_3 \sim 1/\varepsilon_0;$$

thus, the asymptotic series for x_1, x_2 and x_3 formally become:

$$x_1 = x_1^{(0)} + x_1^{(1)} + ...$$
$$x_2 = x_2^{(-1)} + x_2^{(0)} + ...$$
$$x_3 = x_3^{(-1)} + x_3^{(0)} + ...$$

To lowest order we get:

$$\partial_t^2 x_2^{(-1)} = \partial_t^2 x_3^{(-1)} = 0, \tag{5.1}$$

yielding $x_2^{(-1)} = x_2^{(-1)}(\tau)$ and $x_3^{(-1)} = x_3^{(-1)}(\tau)$; *i.e.* x_2 and x_3 to the leading order depend only on the τ variable.

To the next order, equations (3.5) yield:

$$\begin{aligned}\partial_t^2 x_1^{(0)} = -\omega_{\mathrm{p}}^2 x_1^{(0)} + \varepsilon_0 x_1^{(0)}\big[&x_2^{(-1)}(\partial_\tau x_3^{(-1)} + \partial_t x_3^{(0)})\\ &- x_3^{(-1)}(\partial_\tau x_2^{(-1)} + \partial_t x_2^{(0)})\big],\end{aligned} \tag{5.2}$$

$$\partial_t^2 x_2^{(0)} = \varepsilon_0 x_3^{(-1)}(\tau)(\partial_t x_1^{(0)}) x_1^{(0)}, \tag{5.3}$$

$$\partial_t^2 x_3^{(0)} = -\varepsilon_0 x_2^{(-1)}(\tau)(\partial_t x_1^{(0)}) x_1^{(0)}. \tag{5.4}$$

Proceeding further to the next higher order, we may write:

$$\begin{aligned}&\partial_t^2 x_2^{(1)} + 2\partial_t\partial_\tau x_2^{(0)} + \partial_\tau^2 x_2^{(-1)} =\\ &\quad\varepsilon_0 x_3^{(-1)}(x_1^{(0)}\partial_\tau x_1^{(0)} + \partial_t x_1^{(1)} x_1^{(0)} + x_1^{(1)}\partial_t x_1^{(0)})\\ &\quad+ \varepsilon_0 x_3^{(0)}\partial_t x_1^{(0)} x_1^{(0)} - \varepsilon_0 x_1^{(0)2}(\partial_t x_3^{(0)} + \partial_\tau x_3^{(-1)}),\end{aligned} \tag{5.5}$$

$$\partial_t^2 x_3^{(1)} + 2\partial_t\partial_\tau x_3^{(0)} + \partial_\tau^2 x_3^{(-1)} = \\ -\varepsilon_0 x_2^{(-1)}(x_1^{(0)}\partial_\tau x_1^{(0)} + \partial_t x_1^{(1)} x_1^{(0)} + x_1^{(1)}\partial_t x_1^{(0)}) \\ -\varepsilon_0 x_2^{(0)}\partial_t x_1^{(0)} x_1^{(0)} + \varepsilon_0 x_1^{(0)2}(\partial_t x_2^{(0)} + \partial_\tau x_2^{(-1)}), \quad (5.6)$$

Solving equations (5.3, 5.4) we get, respectively:

$$\partial_t x_2^{(0)} = \varepsilon_0 x_3^{(-1)}(\tau)[x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2], \quad (5.7)$$

$$\partial_t x_3^{(0)} = -\varepsilon_0 x_2^{(-1)}(\tau)[x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2], \quad (5.8)$$

which, substituted into equation (5.2), give:

$$\partial_t^2 x_1^{(0)} = x_1^{(0)}[-\omega_\mathrm{p}^2 + \varepsilon_0(x_2^{(-1)}\partial_\tau x_3^{(-1)} - x_3^{(-1)}\partial_\tau x_2^{(-1)}) \\ -\varepsilon_0^2(x_2^{(-1)2}(\tau) + x_3^{(-1)2}(\tau))(x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2)]. \quad (5.9)$$

Here, $\langle\ \rangle$ indicates the average on the fast time scale.

Equation (5.9) can be closed with equations for the time evolution of $x_2^{(-1)}$ and $x_3^{(-1)}$. These are obtained from equations (5.5, 5.6), averaging on the fast time scale, *i.e.*,

$$\partial_\tau^2 x_2^{(-1)} = \varepsilon_0 x_3^{(-1)}\frac{1}{2}\partial_\tau\langle x_1^{(0)2}\rangle - \frac{3}{2}\varepsilon_0\langle x_1^{(0)2}\partial_t x_3^{(0)}\rangle \\ -\varepsilon_0\partial_\tau x_3^{(-1)}\langle x_1^{(0)2}\rangle \quad (5.10)$$

$$\partial_\tau^2 x_3^{(-1)} = -\varepsilon_0 x_2^{(-1)}\frac{1}{2}\partial_\tau\langle x_1^{(0)2}\rangle + \frac{3}{2}\varepsilon_0\langle x_1^{(0)2}\partial_t x_2^{(0)}\rangle \\ +\varepsilon_0\partial_\tau x_2^{(-1)}\langle x_1^{(0)2}\rangle. \quad (5.11)$$

In order to make further progress in the analysis of equations (5.9–5.11), we define the function:

$$F = x_2^2 + x_3^2. \quad (5.12)$$

To leading order, the F function, defined in equation (5.12), is a function of the τ variable only: $F^{(-2)}(t,\tau) = F^{(-2)}(\tau)$; for simplicity we refer to $F^{(-2)}(\tau)$ as $F(\tau)$.

Expanding M_1 to the leading order, we find the expression:

$$M_1^{(-1)} = x_2^{(-1)}\partial_\tau x_3^{(-1)} - x_3^{(-1)}\partial_\tau x_2^{(-1)} \\ -\varepsilon_0 F(\tau)(x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2). \quad (5.13)$$

From equations (3.5), we get the time evolution equations for M_1 and F, respectively:

$$\dot{M}_1 = \varepsilon_0\left(\frac{1}{2}x_1^2\dot{F} - x_1\dot{x}_1 F\right), \quad (5.14)$$

$$\ddot{F} = 2(\dot{x}_2^2 + \dot{x}_3^2) - 2\varepsilon_0 x_1^2 M_1. \quad (5.15)$$

Using equations (5.12, 5.13), in general, equations (5.10, 5.11) can be put into the form:

$$\partial_\tau\langle M_1^{(-1)}\rangle = -\varepsilon_0 F\frac{1}{2}\partial_\tau\langle x_1^{(0)2}\rangle + \frac{1}{2}\langle x_1^{(0)2}\rangle\varepsilon_0\partial_\tau F, \quad (5.16)$$

which can be immediately derived also from equation (5.14), and

$$\partial_\tau((\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2) = -\varepsilon_0\langle M_1^{(-1)}\rangle\partial_\tau\langle x_1^{(0)2}\rangle \\ +\frac{3}{4}\varepsilon_0^2\partial_\tau F(\langle x_1^{(0)4}\rangle - \langle x_1^{(0)2}\rangle^2). \quad (5.17)$$

Furthermore, equation (5.9) can be straightforwardly integrated once and it yields:

$$(\partial_t x_1^{(0)})^2 = a(\tau)x_1^{(0)2} - b(\tau)x_1^{(0)4} + c(\tau), \quad (5.18)$$

where $c(\tau)$ is an integration constant which can be obtained from energy conservation equation (3.6) averaged over the fast time scale;

$$c(\tau) = 2E - (\varepsilon_0^2/4)F(\tau)\langle x_1^{(0)2}\rangle^2 - (\partial_\tau x_2^{(-1)})^2 - (\partial_\tau x_3^{(-1)})^2 \\ -\varepsilon_0(x_2^{(-1)}\partial_\tau x_3^{(-1)} - x_3^{(-1)}\partial_\tau x_2^{(-1)})\langle x_1^{(0)2}\rangle. \quad (5.19)$$

Meanwhile, the expressions for $a(\tau)$ and $b(\tau)$ are the following:

$$a(\tau) = -\omega_\mathrm{p}^2 + \varepsilon_0(x_2^{(-1)}\partial_\tau x_3^{(-1)} - x_3^{(-1)}\partial_\tau x_2^{(-1)}) \\ +(\varepsilon_0^2/2)F(\tau)\langle x_1^{(0)2}\rangle, \quad (5.20)$$

$$b(\tau) = (\varepsilon_0^2/4)F(\tau). \quad (5.21)$$

We can interpret equation (5.18) as the energy conservation law for a particle which moves in an effective potential:

$$V_\mathrm{eff} = (1/2)(-a(\tau)x_1^{(0)2} + b(\tau)x_1^{(0)4}), \quad (5.22)$$

with total energy :

$$E_\mathrm{eff}(\tau) = c(\tau)/2. \quad (5.23)$$

The potential defined in equation (5.22) is a polynomial function of x_1, whose coefficients are not constant but depend on the slow time scale τ; such a potential modifies its shape during the time evolution over the slow scale of the system. From equation (5.18), we see that if $a \leq 0$, $c(\tau)$ must be positive because $b(\tau)$ is positive by definition; only when $a > 0$ can $c(\tau)$ become negative.

When $a \leq 0$, the effective potential admits only one stable equilibrium point, which corresponds to the solutions with zero mean value over the fast time scale:

$$\langle x_1^{(0)}\rangle = 0. \quad (5.24)$$

This situation corresponds to the Abelian phase of the system. In this phase, the solution $x_1^{(0)}$ is quasi-harmonic and satisfies equation (5.24). If we study the phase space diagrams of the system (see Fig. 4), we find that, in the Abelian phase the orbits are ellipses centered in the origin, which correspond to harmonic solutions for $x_1^{(0)}$. During the evolution over the τ scale, the ellipses modify their shape, tending to a two-lobes shape (spectacles-like);

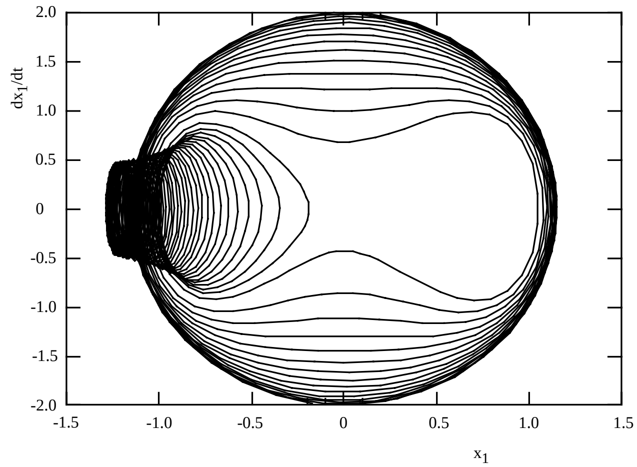

Fig. 4. Phase space of the function x_1, showing the transition from the symmetric orbits of the Abelian phase to asymmetric non-Abelian oscillations.

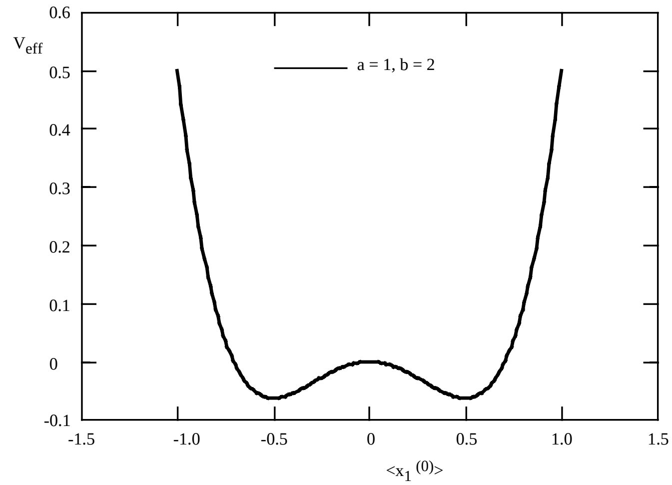

Fig. 5. The effective potential $V_{\rm eff}$ in the non-Abelian phase, showing the existence of two minima for $\langle x_1^{(0)}\rangle \neq 0$.

in particular, the transition from the Abelian to the non-Abelian phase corresponds to a limit 8-shaped orbit with two separate lobes connected at the origin of the axes. Once the transition has taken place, the system chooses one of the two lobes and the orbit becomes smaller and centered at a non zero value, which is the average of x_1 on the fast time scale. The transition can be interpreted in terms of shape modification of the effective potential. As we have previously seen, the condition $a \leq 0$ corresponds to the Abelian phase of the system, and during the transition from the Abelian to the non-Abelian phase the function $a(\tau)$ grows, becoming positive and still increasing. After the transition, when $\langle M_1^{(-1)}\rangle$ saturates to the constant value $\omega_{\rm p}^2/\varepsilon_0$, the function $a(\tau)$ grows as:

$$a(\tau) = \frac{\varepsilon_0^2}{2} F(\tau)\langle x_1^{(0)2}\rangle. \tag{5.25}$$

When $a > 0$ the effective potential assumes the shape shown in Figure 5. Such a potential admits two stable equilibrium points, about which orbits are possible, which correspond to solutions having non-zero mean value on the fast time scale:

$$\langle x_1^{(0)}\rangle = \pm\sqrt{\frac{a}{2b}}\,. \tag{5.26}$$

When $a > 0$, $c(\tau)$ can be negative. In the situation in which the effective potential is negative and the kinetic energy is smaller than $|V_{\rm eff}|$, the motion is confined within one of the two potential wells corresponding to the minima of $V_{\rm eff}$, as expected from the fact that $E_{\rm eff}(\tau)$, as defined in equation (5.23), is also negative. This situation is exactly that of the non-Abelian phase.

Further details on the analytic solutions of equations (3.5) are given in Appendix A, where the procedure for solving equations (5.10, 5.11, 5.18) will be discussed.

6 Discussion: Equations for M_1 and F and heuristic considerations

In the present section, by using the multiple scale analytical approach [11] and some heuristic considerations (such as those concerning the saturation of M_1 and discussed in the previous section), we show that, in the limit of small non-Abelian coupling, $\varepsilon_0 \ll 1$, it is possible to give accurate predictions of the behavior of the system on slow time-scales.

In particular, we will study in some detail the equations for the evolution of the functions M_1 and F, finding analytic expressions, which are valid in the pure Abelian and in the pure non-Abelian phases. It is not easy to follow analytically the system around the transition, but, if one assumes that it happens almost instantaneously, which is confirmed by the numerical simulations for the cases we have considered, it is possible to give accurate estimates of characteristic times, such as the transition time and the duration of the non-Abelian phase. This is important in order to show that the main physical behaviour of the system can be predicted analytically, without having to perform numerical simulations for every choice of the parameters.

Using equation (5.14) it is easily shown that, to leading order, the time evolution of M_1 is given by:

$$\partial_t M_1^{(-1)} = -\varepsilon_0 \partial_t (x_1^{(0)2}/2) F(\tau). \tag{6.1}$$

Integrating equation (6.1), we get:

$$M_1^{(-1)} = -\frac{1}{2}\varepsilon_0 x_1^{(0)2} F(\tau) + M(\tau), \tag{6.2}$$

where $M(\tau)$ is an integration constant which depends only on the τ scale.

Meanwhile, using the definition of $F(\tau)$, equation (5.12), and equations (5.10, 5.11), it is possible to show that the following equation determines the time evolution of $F(\tau)$:

$$\partial_\tau^2 F(\tau) = 2(\partial_\tau x_2^{(-1)2} + \partial_\tau x_3^{(-1)2}) - 2\varepsilon_0 \langle x_1^{(0)2}\rangle \langle M_1^{(-1)}\rangle + 3/2\varepsilon_0^2 F(\tau)(\langle x_1^{(0)4}\rangle - \langle x_1^{(0)2}\rangle^2), \quad (6.3)$$

which can be easily derived also from equation (5.15).

Using the energy conservation law, equation (3.6), along with equations (5.7, 5.8), we obtain:

$$(\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2 + (\varepsilon_0^2/4)F(\tau)(\langle x_1^{(0)4}\rangle - \langle x_1^{(0)2}\rangle^2) = 2E - \omega_{\rm p}^2 \langle x_1^{(0)2}\rangle - \langle \partial_t x_1^{(0)2}\rangle. \quad (6.4)$$

During the pure plasma phase, the solution $x_1^{(0)}$ is an harmonic function:

$$x_1^{(0)} = x_{10}\sin(\omega_{\rm p} t). \quad (6.5)$$

Therefore, from equation (5.16), solved for $\langle x_1^{(0)2}\rangle = const$, we easily find that in the pure Abelian phase the mean value over the fast scale of $M_1^{(-1)}$ is:

$$\langle M_1^{(-1)}\rangle = \frac{\varepsilon_0}{2}\langle x_1^{(0)2}\rangle F(\tau). \quad (6.6)$$

Averaging equation (6.2) over the fast time-scale and using equation (6.6), we find the explicit form of $M(\tau)$. Thus, for $M_1^{(-1)}$, in the Abelian phase, we find the expression:

$$M_1^{(-1)} = \varepsilon_0 F(\tau)(\langle x_1^{(0)2}\rangle - x_1^{(0)2}/2). \quad (6.7)$$

Using equation (6.5), equation (6.4), in the pure Abelian phase, becomes :

$$(\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2 + (\varepsilon_0^2/4)F(\tau)(\langle x_1^{(0)4}\rangle - \langle x_1^{(0)2}\rangle^2) = const = \dot{x}_{20}^2 + \dot{x}_{30}^2. \quad (6.8)$$

Evaluating, according to equation (6.5), the quantity $\langle x_1^{(0)4}\rangle - \langle x_1^{(0)2}\rangle^2$ and substituting equations (6.6, 6.8) into equation (6.3), we finally get, in the Abelian phase:

$$\partial_\tau^2 F = 2J^2 - \omega_{\rm Abel}^2 F(\tau), \quad (6.9)$$

where $J^2 = \dot{x}_{20}^2 + \dot{x}_{30}^2$, and $\omega_{\rm Abel} = \varepsilon_0 x_{10}^2/2\sqrt{2}$.

Equation (6.9) is the equation of an harmonic oscillator driven by a constant force. This equation with the initial condition $F(0) = \dot{F}(0) = 0$, admits the harmonic solution:

$$F(\tau) = \frac{4J^2}{\omega_{\rm Abel}^2}\sin^2\left(\frac{\omega_{\rm Abel}\tau}{2}\right). \quad (6.10)$$

Similar considerations can be made in the truly non-linear non-Abelian phase. At saturation, we found that M_1 has the constant mean value $\langle M_1^{(-1)}\rangle = \omega_{\rm p}^2/\varepsilon_0$, as shown in equation (4.2).

Averaging equation (6.2) over the fast time scale and using equation (4.2), we find the explicit form for $M_1^{(-1)}$ in the pure non-Abelian phase:

$$M_1^{(-1)} = \frac{\omega_{\rm p}^2}{\varepsilon_0} - \varepsilon_0 F(\tau)(x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2). \quad (6.11)$$

Since, at saturation, $M_1^{(-1)}$ acquires a mean value which is constant with respect to the τ scale, equation (5.16) may be used to show that:

$$\frac{\partial_\tau F}{F} = \frac{\partial_\tau\langle x_1^{(0)2}\rangle}{\langle x_1^{(0)2}\rangle}, \quad (6.12)$$

which implies:

$$\langle x_1^{(0)2}\rangle = kF, \quad (6.13)$$

where k is a constant. Thus, the mean value of $x_1^{(0)2}$ is proportional to $F(\tau)$ during the non-Abelian phase. As $\langle x_1^{(0)2}\rangle$ is of order 1, and F is of order $O(\varepsilon_0^{-2})$, it follows from equation (6.13) that k is of order $O(\varepsilon_0^2)$. Furthermore, anticipating the numerical results of Section 7, we observe that the solution x_1, during the non-Abelian phase, acquires a mean value which is large with respect to the amplitude of the rapidly oscillating part of the function. So we can assume that $x_1^{(0)}$ coincides with its mean value except for small terms, rapidly oscillating over the fast scale. In formulae:

$$x_1^{(0)} = \langle x_1^{(0)}\rangle + \delta x_1, \quad (6.14)$$

where δx_1 is an oscillating function for which we assume the expression:

$$\delta x_1 = |\delta x_1|\psi(t). \quad (6.15)$$

The amplitude $|\delta x_1|$ is a function of the τ scale only and it is small with respect to $\langle x_1^{(0)}\rangle$. Thus, $\psi(t)$ is a rapidly oscillating function with zero mean value over the τ scale and with unitary amplitude.

Under the assumptions mentioned above:

$$x_1^{(0)} \cong \langle x_1^{(0)}\rangle = \pm\sqrt{\frac{a}{2b}} = \pm\sqrt{kF}, \quad (6.16)$$

$$\langle x_1^{(0)2}\rangle \cong \langle x_1^{(0)}\rangle^2 = \frac{a}{2b} = kF, \quad (6.17)$$

$$\langle x_1^{(0)4}\rangle \cong \langle x_1^{(0)}\rangle^4 = \frac{a^2}{4b^2} = k^2F^2. \quad (6.18)$$

If we plot the function F, defined by equation (5.12), *vs.* time, we find that it has a quite regular shape and it is independent from the rapid fluctuations which correspond to the evolution over the fast scale (see Fig. 6).

Proceeding as it was previously done for the Abelian phase, and using equation (4.2) for M_1 and equations (6.16–6.18) for $x_1^{(0)}$, equation (6.3) during the non-Abelian phase can be written as:

$$\partial_\tau^2 F = 2((\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2) - 2\omega_{\rm p}^2 kF(\tau). \quad (6.19)$$

Calculating $(\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2$ from equation (6.4), and neglecting the time derivative of $x_1^{(0)}$ (according to Eq. (6.14)), we obtain:

$$(\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2 = 2E - \omega_{\rm p}^2 kF(\tau). \quad (6.20)$$

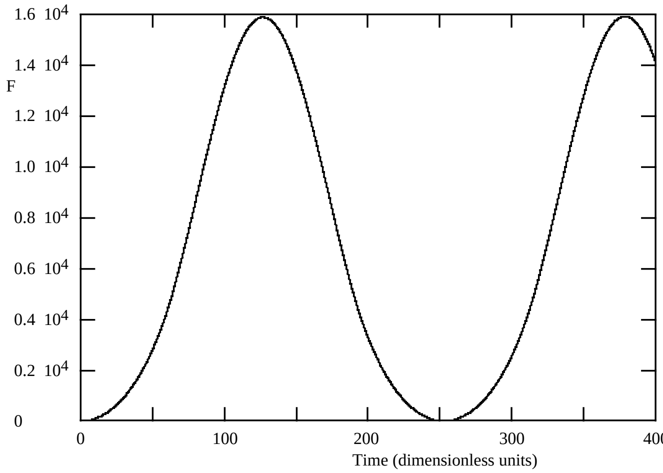

Fig. 6. Time evolution of the function F, showing the absence of fast time-scale features and its approximately sinusoidal behavior.

Thus, the equation governing the time evolution of F during the non-Abelian phase is:

$$\partial_\tau^2 F = 2(2E) - \omega_{\rm sat}^2 F(\tau), \tag{6.21}$$

where $\omega_{\rm sat}^2 = 4\omega_{\rm p}^2 k$, and the subscript "sat" stands for "saturation", which is reached by M_1 during this phase. If we remember that k is of order $O(\varepsilon_0^2)$, it follows that $\omega_{\rm sat}$ is of the same order of $\omega_{\rm Abel}$, and, in fact, in our numerical simulations their values are not very different. We therefore see that the transition to the non-Abelian phase modifies the parameters, but leaves the structure of the equation for the time evolution of F unchanged, this equation being still that of an harmonic oscillator driven by an external constant force. The solution of equation (6.21) can be put into the form:

$$F(\tau) = \frac{2(2E)}{\omega_{\rm sat}^2} + \widetilde{F}\cos(\omega_{\rm sat}\tau + \phi), \tag{6.22}$$

where the amplitude $\widetilde{F}$ and the phase ϕ are fixed by the initial conditions. If we assume $F(\tau_{\max}) = F_{\max}$ and $\dot{F}(\tau_{\max}) = 0$ at the instant $\tau_{\max}$ when F reaches a maximum, we find:

$$F(\tau) = \frac{4(2E)}{\omega_{\rm sat}^2}\sin^2\left(\frac{\omega_{\rm sat}\tau - \omega_{\rm sat}\tau_{\rm sat}}{2}\right) + F_{\max}\cos(\omega_{\rm sat}\tau - \omega_{\rm sat}\tau_{\max}). \tag{6.23}$$

Using equations (5.17, 6.23), we obtain that the following equation holds during non-Abelian phase:

$$(\partial_\tau x_2^{(-1)})^2 + (\partial_\tau x_3^{(-1)})^2 + \omega_{\rm p}^2 k(x_2^{(-1)2} + x_3^{(-1)2}) = const = \omega_{\rm p}^2 k F_{\max}. \tag{6.24}$$

7 Comparison between analytical estimates and numerical results

We performed numerical simulations of equations (3.5). In this section, we discuss in particular two of the cases analyzed also by Bhatt *et al.* [5], labeled (a) and (c) in the following, along with another case (b), with an intermediate choice of parameters and initial conditions:

(a) $\varepsilon_0 = 0.05, \quad x_1(0) = x_2(0) = x_3(0) = 0,$
$\dot{x}_1(0) = 2, \quad \dot{x}_2(0) = 1, \quad \dot{x}_3(0) = 3;$

(b) $\varepsilon_0 = 0.05, \quad x_1(0) = x_2(0) = x_3(0) = 0,$
$\dot{x}_1(0) = 2, \quad \dot{x}_2(0) = \sqrt{0.1}, \quad \dot{x}_3(0) = \sqrt{0.9};$

(c) $\varepsilon_0 = 0.05, \quad x_1(0) = x_2(0) = x_3(0) = 0,$
$\dot{x}_1(0) = 2, \quad \dot{x}_2(0) = 0.1, \quad \dot{x}_3(0) = 0.3.$

We reproduced the numerical results shown in Figures 1 and 3 of Bhatt *et al.* [5], except for the trivial fact that in their paper the labels of the two figures have been accidentally exchanged (see Fig. 7). If one looks at these simulations, the solution for x_1 is quasi-harmonic before the transition, with a slowly decreasing frequency. That this interpretation is indeed correct is rigorously shown in Appendix A.

If we assume that the transition to the non-Abelian phase takes place instantaneously at a time $\tau_{\rm sat}$, and if the transition time satisfies the condition:

$$\omega_{\rm Abel}\tau_{\rm sat} \ll 1, \tag{7.1}$$

we can also assume that the functional form of M_1 and F until the transition time is that of the pure plasma phase, respectively given by equations (6.7, 6.10), with the frequency $\omega_{\rm Abel}$ replaced by a frequency $\omega_{\rm bt}(\tau)$ ("bt" stands for "before transition") which is a slowly varying function evolving from the initial value $\omega_{\rm Abel}$, calculated in the pure Abelian phase, $\omega_{\rm Abel} = \varepsilon_0 x_{10}^2/2\sqrt{2}$, to the final value $\omega_{\rm sat} = 2\omega_{\rm p}\sqrt{k}$, of the non-Abelian phase. Thus:

$$F_{\rm bt}(\tau) = \frac{4J^2}{\omega_{\rm bt}^2(\tau)}\sin^2\left(\frac{\omega_{\rm bt}(\tau)\tau}{2}\right). \tag{7.2}$$

In our numerical simulations (a), (b) and (c) we did not change the value of x_{10}, so $\omega_{\rm Abel}$ was constant. Furthermore, the values of k turned out to be approximately the same (within 10%); as a consequence, the values of $\omega_{\rm sat}$ were very similar to each other, and approximately equal to $1.4\omega_{\rm Abel}$. This is an interesting result, because $\omega_{\rm Abel}$ is immediately computed from the input parameters, while $\omega_{\rm sat}$ is dependent on the value of k, which should be found from the output of the numerical simulations. If we remember that the duration of the non-Abelian phase is dependent on $\omega_{\rm sat}$, this relation allows us to predict this important physical quantity without performing a numerical simulation for every choice of the parameters.

By using equations (4.2, 6.7, 7.2), we obtain a condition on $\tau_{\rm sat}$ by imposing that the Abelian expression of M_1 saturates at the transition time to the constant value it maintains during the non-Abelian phase:

$$\langle M_1^{(-1)}\rangle_{\rm sat} = \frac{\omega_{\rm p}^2}{\varepsilon_0} = \frac{\varepsilon_0}{2}\langle x_1^{(0)2}\rangle F_{\rm bt}(\tau_{\rm sat}). \tag{7.3}$$

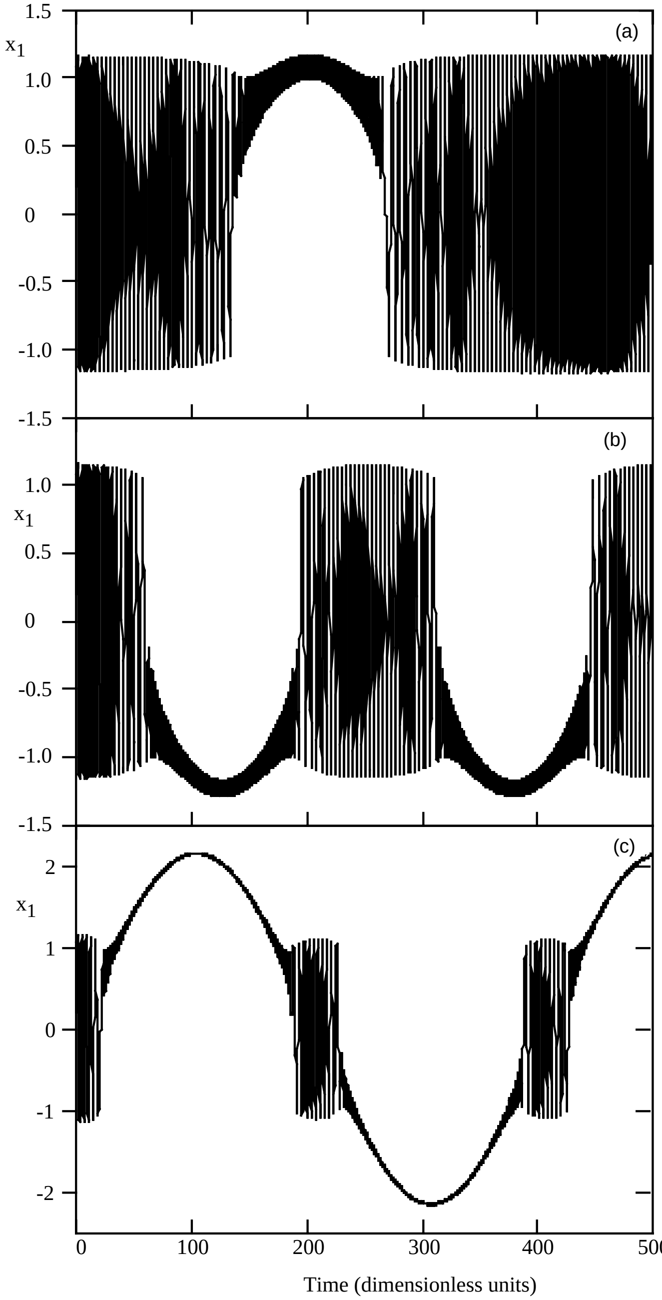

Fig. 7. Time evolution of the function x_1, for the three sets of parameters (a), (b) and (c) of our numerical simulations.

This equation implies also that the value of F at the transition be the same for all the cases (a), (b) and (c), as it is verified by the numerical solutions.

With the approximation $\langle x_1^{(0)2}\rangle \approx x_{10}^2/2$ and expanding the function F for $\omega_{\rm bt}\tau_{\rm sat} \ll 1$, we can roughly estimate the saturation time as:

$$\tau_{\rm sat}^2 \approx \frac{4\omega_{\rm p}^2}{J^2 x_{10}^2 \varepsilon_{10}^2}, \tag{7.4}$$

obtaining the estimates :

$$\tau_{\rm sat} = 19; \quad \tau_{\rm num.sim.} = 20.5,$$
$$\tau_{\rm sat} = 60; \quad \tau_{\rm num.sim.} = 59.5,$$

which are in good agreement with the numerical simulations for the cases (a) and (b), for which the condition (7.1) is approximately satisfied. When the condition (7.1) does not hold, as in case (c), the function F is strongly distorted from a sinusoidal shape due to the growth of J^2 on a time-scale which is of the same order of the period of F.

During the saturation phase, let us assume the following expression for $M_1^{(-1)}(t,\tau)$:

$$M_1^{(-1)}(t,\tau) = \frac{\omega_{\rm p}^2}{\varepsilon_0}(1-\psi(t)) \tag{7.5}$$

where, again, $\psi(t)$ is a rapidly oscillating function with zero mean value over the τ scale and with unitary amplitude.

Using equations (6.14, 6.16) we get:

$$x_1^{(0)2} - \langle x_1^{(0)2}\rangle \sim 2\langle x_1^{(0)}\rangle |\delta x_1| \psi(t). \tag{7.6}$$

Comparing equation (6.11) with equation (7.5), we find:

$$\varepsilon_0 F(\tau)(x_1^{(0)2}/2 - \langle x_1^{(0)2}\rangle/2) = \frac{\omega_{\rm p}^2}{\varepsilon_0}\psi(t). \tag{7.7}$$

Therefore, substituting equation (7.6) into equation (7.7), we get :

$$|\delta x_1| = \frac{\omega_{\rm p}^2}{\varepsilon_0^2 \langle x_1^{(0)}\rangle F}\cdot \tag{7.8}$$

In this way, we have obtained a relation between $\langle x_1^{(0)}\rangle$, F and the amplitude of the rapidly oscillating part of x_1 during the non-Abelian phase. Substituting the numerical values of F and $\langle x_1^{(0)}\rangle$ for all the cases (a), (b) and (c) we get an excellent agreement with the results of our numerical simulations:

(a) $|\delta x_1| = 0.0098; \quad |\delta x_1|_{\rm num.sim.} = 0.0087,$
(b) $|\delta x_1| = 0.062; \quad |\delta x_1|_{\rm num.sim.} = 0.061,$
(c) $|\delta x_1| = 0.091; \quad |\delta x_1|_{\rm num.sim.} = 0.090.$

8 Numerical simulations of a SU(3) plasma

We have performed numerical simulations also in the case of longitudinal 1D stationary waves in a classical SU(3) Yang-Mills plasma. The system of non-linear equations we obtained, written in terms of rescaled gauge fields and of the adimensional variable $t = \omega_{\rm p}\xi/\beta$, with the same definitions of β, $\omega_{\rm p}$ and ε given by Bhatt *et al.* [5] in the case of an SU(2) plasma, is of the form:

$$\ddot{a}_a = -\left(\sum_b \delta_{ab} + \varepsilon f_{abc} a_b \dot{a}_c\right) \sum_d a_d. \tag{8.1}$$

Here the indexes run from 1 to 8 because in the case of SU(3) there are eight infinitesimal generators of group transformations and consequently eight gauge fields; f_{abc} represent structure constants of the SU(3) group defined by the commutation rules:

$$[\lambda_i, \lambda_j] = i f_{ijk} \lambda_k, \tag{8.2}$$

and λ_i are the Gell-Mann matrices which represent the SU(3) infinitesimal generators.

The analogue of system (3.5) in the SU(3) case is:

$$\begin{cases} \ddot{y}_1 = -\left(\omega_{\mathrm{p}}^2 - \dfrac{\varepsilon}{\sqrt{8}} N_1\right) y_1 \\ \ddot{y}_{i\neq 1} = \dfrac{\varepsilon}{\sqrt{8}} N_{i\neq 1} y_1 \end{cases} \tag{8.3}$$

where:

$$N_i = f_{ijk} y_j \dot{y}_k, \tag{8.4}$$

the plasma frequency is, in dimensionless variables $\omega_{\mathrm{p}} = \sqrt{8}$, and the definitions of the y_i are, in analogy with the SU(2) case:

$$\begin{cases} y_1 = (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8) \\ y_2 = (2/\sqrt{21})(a_1 - 6a_3 + a_4 + a_5 + a_6 + a_7 + a_8) \\ y_3 = (1/\sqrt{7})(a_1 - 7a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8) \\ y_4 = (2/\sqrt{15})(a_1 - 5a_4 + a_5 + a_6 + a_7 + a_8) \\ y_5 = \sqrt{2/5}(a_1 - 4a_5 + a_6 + a_7 + a_8) \\ y_6 = \sqrt{2/3}(a_1 - 3a_6 + a_7 + a_8) \\ y_7 = (2/\sqrt{3})(a_1 - 2a_7 + a_8) \\ y_8 = 2(a_1 - a_8) \end{cases} . \tag{8.5}$$

In Figure 8 we show the result of our numerical simulations, obtained with a standard Runge Kutta method, for the variable y_1, which is the SU(3) analogue of the variable x_1 previously defined in the case of SU(2), as N_1 is the SU(3) analogue of the variable M_1.

The numerical simulation of Figure 8 was performed with the following choice of parameters:

$$\varepsilon = 0.05\sqrt{3}, \quad y_i = 0, \quad \dot{y}_1 = 0.2,$$
$$\dot{y}_2 = \dot{y}_3 = \dot{y}_4 = \dot{y}_5 = \dot{y}_6 = \dot{y}_7 = 0.01,$$
$$\dot{y}_8 = 0.1.$$

The simulations show that the system, after an Abelian phase with fast plasma oscillations, experiences a transition to a non-Abelian phase, characterized by much slower oscillations on which small amplitude fast oscillations are initially superimposed. The amplitude of these small oscillations vanishes rapidly so that $y_1 = \langle y_1 \rangle$, and the system does not show any sign of returning to the Abelian behavior (see Fig. 8). Of course, this non-Abelian phase is a decoupling one, as it can also be seen from Figure 9, showing the behavior of the quantity N_1, which, during the non-Abelian phase, saturates to a constant value:

$$\left\langle N_1 \right\rangle = \frac{\omega_{\mathrm{p}}^2 \sqrt{8}}{\varepsilon} \, . \tag{8.6}$$

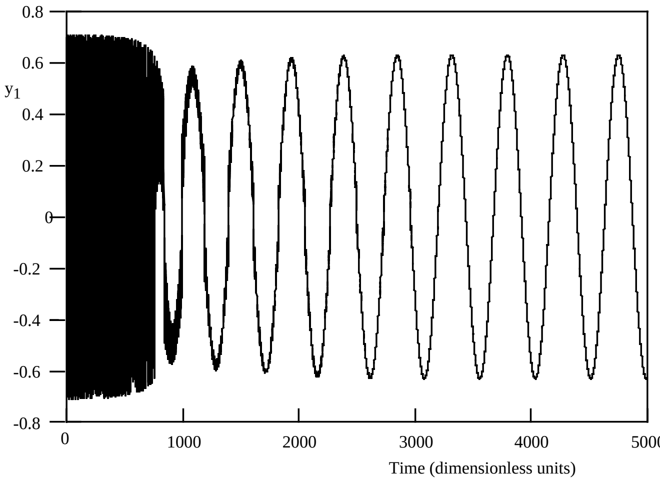

Fig. 8. Time evolution of the function y_1, representing the fluid velocity of the light quarks of a two component SU(3) plasma, showing the transition to a slowly oscillating non-Abelian phase.

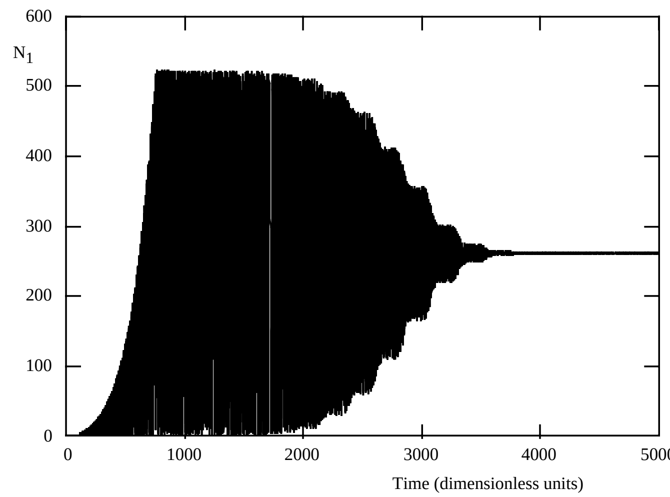

Fig. 9. Time evolution of the function N_1, showing the saturation of $\langle N_1 \rangle$ at the transition to the non-Abelian phase, as it happens for M_1 in the SU(2) case.

This saturation condition is equivalent to the decoupling condition :

$$\left\langle \sum_j^8 I_j \right\rangle = 0 \tag{8.7}$$

in complete analogy with the SU(2) case.

The decoupling effect is clearly visible in Figure 10, showing the dramatic increase of the physical space (the integral of y_1) accessible to the system in the non-Abelian phase.

9 Conclusions and outlook

In the present paper we have discussed an analytical approach to the non linear system (3.5) that, under certain

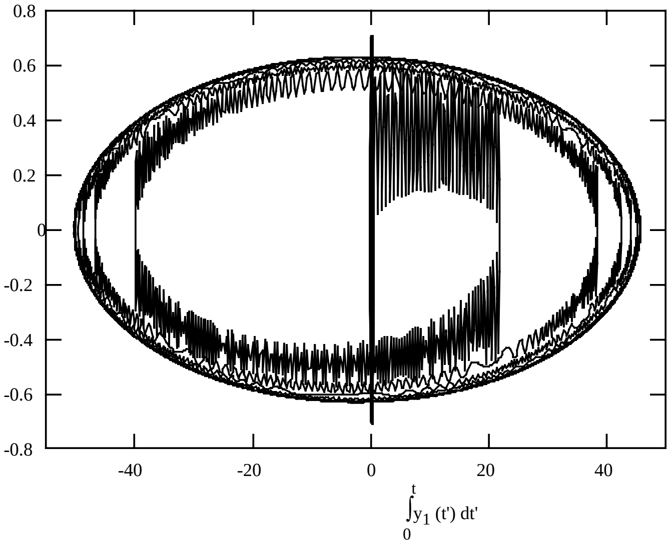

Fig. 10. Increase of the physical space available to the light quarks (the integral of the fluid velocity y_1) after the non-Abelian transition. The squeezed vertical ellipse is the Abelian phase orbit, the large ellipse is that of the limit-cycle slow-frequency non-Abelian oscillations.

perturbative conditions, allows us to make quantitative predictions about relevant quantities for the observed non-linear oscillations, such as the transition time from the Abelian to the non-Abelian phase and the duration of the two phases. We have found the conditions for which this transition occurs and a physical interpretation of the non-Abelian phase as a decoupling one, in which light quarks have a drift motion with a velocity that is approximately constant, over the time scale of ordinary U(1) plasma oscillations. In other words, during the non-Abelian phase, light quarks seem to spend a time, which is large with respect to the fast oscillation time-scale, in a quasi-free state that allows them to slide away with respect to the heavy quarks which constitute the background. This is an interesting property of the SU(2) longitudinal 1D plasma oscillations, related to the precession of the color charge vector in color space. If this is confirmed as a general property of non-Abelian plasmas, in other words if it is also a property of QCD SU(3) plasmas, as it seems indeed to be from our numerical simulations, we might conjecture that color screening is indeed effective in a quark gluon plasma and use this idea as a starting point for further and more realistic studies, with the aim of interpreting experimental observations, *e.g.* in relativistic heavy ion collisions [2].

As mentioned in the Introduction, the time scale of precession of the color charge vector in color space – yielding the dynamically decoupled phase on the fast time-scale of plasma oscillations, as discussed in the present work, both in the SU(2) and the SU(3) case – is comparable with the time scale associated with the thermal gluon induced plasmon damping rate, discussed in reference [8]. Furthermore, we must recall that our present results are obtained in the cold plasma approximation. Here, we want to stress that, even though this limit is not "strictly" verified in physically relevant situations, the characteristic times for the onset of non-Abelian effects (our slow time scale) are of the same order or smaller than the computed characteristic plasmon damping times for typical parameters.

For example, let us compute our slow time scale for $m = 1.0$ GeV, $\varepsilon = 0.5$ (*i.e.*, asymptotic expansion parameter $\varepsilon_0 = 0.289$) and $\omega_\mathrm{p} = \varepsilon m/I_0 = 1.0$ GeV (*i.e.*, $I_0 = 1/2$); yielding $\tau_\mathrm{s} = 1/\varepsilon\omega_\mathrm{p} = 0.394$ fm/c.

This value is of the same order of the thermalization time $\tau_\mathrm{th} = 0.5$ fm/c, as computed for $T = 300$ MeV by Muller and Trayanov [11] and of the thermalization time $\tau_\mathrm{th} = 0.3$ fm/c estimated by Shuryak [12] at the Brookhaven Relativistic Heavy Ion Collider energies, as well as of the damping time for plasmons $\tau_\mathrm{th} = 0.3$ fm/c at $T = 300$ MeV, as computed by Braaten and Pisarski [8].

In the view of these results, in our opinion, both non-Abelian dynamics effects and plasmon thermalization processes should to be taken into account for estimating the time behavior of plasma oscillations in a quark-gluon plasma. As mentioned in Section 1, this could be done within the framework of multiple time scale asymptotic expansions, which, due to the results of Section 8, seem to be particularly appropriate in the case of SU(3).

Appendix A

In this appendix, we further delineate the analytic solutions of equation (3.5), already discussed in Section 5.

There, it was qualitatively demonstrated that $c(\tau) > 0$ (*i.e.* $E_\mathrm{eff} > 0$) corresponds to the "plasma phase" whereas the "non-Abelian phase" is reached when $c(\tau) < 0$ (*i.e.* $E_\mathrm{eff} < 0$). Furthermore, it was also shown that, for $a(\tau) \leq 0$, $c(\tau)$ is forced to be positive, *i.e.*, only the plasma phase is possible. To see this fact more clearly, let us derive, from equation (4.18), the expression of the $x_1^{(0)}$ mean value and of the period of oscillation over the fast scale T_fast. We have:

$$\langle x_1^{(0)} \rangle = \frac{1}{T_\mathrm{fast}} \oint \frac{x_1^{(0)} \mathrm{d}x_1^{(0)}}{\sqrt{c + a x_1^{(0)2} - b x_1^{(0)4}}}, \tag{A.1}$$

$$T_\mathrm{fast} = \oint \frac{\mathrm{d}x_1^{(0)}}{\sqrt{c + a x_1^{(0)2} - b x_1^{(0)4}}} \cdot \tag{A.2}$$

If we calculate the integral of equation (A.1) in the Abelian phase, *i.e.* for $c > 0$, we clearly get $\langle x_1^{(0)} \rangle = 0$ from symmetry considerations; whereas in the non-Abelian phase, with the condition $c < 0$ we get:

$$\langle x_1^{(0)} \rangle = \frac{\pi}{T_\mathrm{fast} b^{1/2}}, \tag{A.3}$$

while for the mean value of $x_1^{(0)3}$ in the same conditions we obtain:

$$\langle x_1^{(0)3} \rangle = \frac{a}{2b} \frac{\pi}{T_\mathrm{fast} b^{1/2}} \cdot \tag{A.4}$$

The calculated mean values satisfy the identity (which is readily interpreted by minimizing the effective potential V_{eff}, introduced in Sect. 5, Eq. (5.22)):

$$a\langle x_1^{(0)}\rangle - 2b\langle x_1^{(0)3}\rangle = 0. \tag{A.5}$$

One can also show that:

$$\left(a\langle x_1^{(0)2}\rangle - b\langle x_1^{(0)4}\rangle + c\right) = \frac{1}{T_{\text{fast}}}\oint \mathrm{d}x_1^{(0)}\sqrt{ax_1^{(0)2} - bx_1^{(0)4} + c}, \tag{A.6}$$

from which the following identity is derived:

$$a\langle x_1^{(0)2}\rangle - \frac{3}{2}b\langle x_1^{(0)4}\rangle + \frac{c}{2} = 0 \tag{A.7}$$

Comparing (A.7) with equation (5.18) averaged over the fast time scale, we obtain:

$$\langle(\partial_t x_1^{(0)})^2\rangle = 2b\langle x_1^{(0)4}\rangle - a\langle x_1^{(0)2}\rangle. \tag{A.8}$$

Equation (A.8) shows explicitly why, in the non-Abelian phase, where $a(\tau)$ must be positive, the amplitude of fast oscillations is greatly reduced.

Let us now calculate the oscillation period of the fast oscillations from equation (A.2) in both Abelian ($c > 0$) and non-Abelian ($c < 0$) phases. Reducing equation (A.2) to elliptic integrals we obtain, for $c > 0$:

$$T_{\text{fast}} = \frac{4}{\sqrt[4]{a^2 + 4bc}}K(\zeta), \tag{A.9}$$

with:

$$\zeta = \sqrt{\frac{1}{2} + \frac{a/4b}{\sqrt{a^2/4b^2 + c/b}}}, \tag{A.10}$$

and in the case $c < 0$:

$$T_{\text{fast}} = \sqrt{\frac{4}{a/2 + \sqrt{bc + a^2/4}}}K(\chi), \tag{A.11}$$

with:

$$\chi = \frac{1}{\sqrt{\frac{1}{2} + \frac{a/4b}{\sqrt{a^2/4b^2 + c/b}}}}, \tag{A.12}$$

where K is the complete elliptic integral of the first kind.

Similarly, if we compute $\langle x_1^{(0)2}\rangle$, we obtain, for $c < 0$:

$$\langle x_1^{(0)2}\rangle = \left(\frac{a}{2b} + \sqrt{\frac{a^2}{4b^2} + \frac{c}{b}}\right)\frac{E(\chi)}{K(\chi)}, \tag{A.13}$$

where $E(\chi)$ is the complete elliptic integral of the second kind; and, for $c > 0$:

$$\langle x_1^{(0)2}\rangle = \frac{a}{2b} - \sqrt{\frac{a^2}{4b^2} + \frac{c}{b}}\left(1 - 2\frac{E(\zeta)}{K(\zeta)}\right). \tag{A.14}$$

We observe that the expression for $\langle x_1^{(0)2}\rangle$ is actually a transcendental equation for $\langle x_1^{(0)2}\rangle$ since $a(\tau)$ itself depends on $\langle x_1^{(0)2}\rangle$.

Finally, for $\langle x_1^{(0)4}\rangle$, we get:

$$\langle x_1^{(0)4}\rangle = \frac{1}{3}\left[\left(\frac{a^2}{b^2} + \frac{c}{b} - \frac{2a}{b}\sqrt{\frac{a^2}{4b^2} + \frac{c}{b}}\right) + \frac{4a}{b}\sqrt{\frac{a^2}{4b^2} + \frac{c}{b}}\frac{E(\zeta)}{K(\zeta)}\right], \tag{A.15}$$

for $c > 0$ and, for $c < 0$:

$$\langle x_1^{(0)4}\rangle = \frac{1}{3}\left[\left(\frac{a^2}{b^2} + \frac{2a}{b}\sqrt{\frac{a^2}{4b^2} + \frac{c}{b}}\right)\frac{E(\chi)}{K(\chi)} + \frac{c}{b}\right]. \tag{A.16}$$

Equations (A.9–A.16) entirely determine the period of the fast oscillations, T_{fast}, and the averaged quantities $\langle x_1^{(0)}\rangle$, $\langle x_1^{(0)2}\rangle$, $\langle x_1^{(0)3}\rangle$ and $\langle x_1^{(0)4}\rangle$, as functions of the slow time variable τ. In general, explicit solutions of these transcendental equations cannot be written. However, in Section 6 this has been done under quite general assumptions. Furthermore, from the expression of T_{fast} in the Abelian phase, equation (A.9), it can be shown that T_{fast} increases as $c \to 0$, *i.e.* as the transition to the non-Abelian phase is approached.

After T_{fast} and $\langle x_1^{(0)n}\rangle$ ($n = 1, 2, 3, 4$) are known, these may be substituted back into equations (5.10, 5.11) and used to determine $x_2^{(-1)}(\tau)$ and $x_3^{(-1)}(\tau)$, *i.e.*, the complete behavior of the non linear oscillations to the lowest order.

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